AL67-760

Translation Number

C 402

COPY
No. ____

118

ADDITIONAL COPIES AVAILABLE AT REDUCED RATES FROM

TECHNICAL
MFD
INC
TRANSLATIONS

A TRANSLATION BY

MORRIS D. FRIEDMAN, INC.

Offices at
1383A WASHINGTON STREET

Mail to
P.O. BOX 35

WEST NEWTON 65, MASSACHUSETTS

Telephone
WOODWARD 9-8918 - 8919

(NASA-TT-85760) CORRELATION OF THE AMPLITUDE AND PHASE FLUCTUATIONS IN WAVE PROPAGATION IN MEDIA WITH RANDOM INHOMOGENEITIES (NASA) 11 p

N68-70763

Unclas
00/32 0140222

MORRIS D. FRIEDMAN - Two Pine Street - West Concord, Massachusetts, U.S.A.

Russian Translation

Correlation of the Amplitude and Phase Fluctuations in Wave Propagation in Media with Random Inhomogeneities

L. A. CHERNOV

7N-32-TM
140222

Akusticheskii Zhurnal, vol. 1, No. 1, 1955, pp. 89-95

Abstract: The correlation coefficient is calculated for amplitude and phase fluctuations at the receiver. It is shown that in the case of rough-scale inhomogeneities, the autocorrelation between the amplitude (or phase) fluctuations at various receiver points extends over a distance of the same order as the correlation between the fluctuations of the index of refraction in the medium.

Apparently, non-regular time-and-space variations of the properties of the medium are observed, regularly, in all real media. Random inhomogeneities scatter waves as they are propagated. The scattered waves, superimposed on the primary wave, cause fluctuations in the amplitude and phase of the resultant field.

For example, such phenomena as fluctuation in loudness in acoustics, the twinkling of stars in optics, occasional fading in radio engineering depend on the influence of the random inhomogeneities of the medium. From this brief list of phenomena, referring to various branches of physics, there follows that the question of the fluctuations of the basic characteristics of wave fields is one of the general questions of wave propagation theory.

A dependence must exist between the fluctuations of the index of refraction of the medium and the fluctuations of the characteristics of the wave field. The problem is to establish this dependence. Using this dependence, conclusions can be made on the statistical properties of the wave field; to know the statistical properties of the medium and conversely. Therefore, the study of fluctuations in waves, interesting in itself as it is, opens new possibilities of studying the properties of the medium through which the wave passes.

Works by a number of Soviet [1,2] and foreign [4,5] authors are devoted to the computation of the amplitude and phase fluctuation. However, the theoretical and practical question of interest on the correlation of amplitude and phase fluctuations has been studied but slightly. Here, two questions naturally arise:

1. Does a correlation exist between the amplitude and phase fluctuations at the receiver?

2. Does an auto-correlation exist between the amplitude (or phase) fluctuations at various receiver points and what is the extent of this auto-correlation?

Insofar as we know, the first question has not yet been answered. The second question was considered by certain authors [1,4] only in the ray approximation (geometric optics). Here, however, still not apparent is the simple rule that the auto-correlation between the amplitude (or phase) fluctuations extends approximately the same distance as the correlation between the random inhomogeneities of the medium itself, if the scale of the latter is large compared with the wave length.

Both questions are analyzed below under the assumptions that the medium is isotropic, that there are no regular variations of homogeneity, that the random deviations of the index of refraction of the medium from the average value are small and that their scale is large compared with the wave length.

Let us derive the initial formulas for the subsequent investigation by reproducing (with slight variations) the corresponding part of the work of A. M. Obukhov [2].

Let the index of refraction oscillate around the average value equal to unity:

(1) $$n = 1 + \mu \, ; \qquad |\mu| \ll 1$$

Here μ depends on the coordinates. We will not take the time-dependence of μ explicitly into account, but consider that the characteristics of the medium vary very slowly with time (the frequency of the variation is small in comparison to the wave frequency). Then the wave function Ψ , characterizing the wave field, will satisfy the equation:

$$\nabla^2\psi - \frac{(1+\mu)^2}{c_0^2}\psi = 0 \tag{2}$$

where c_o is the average value of the speed of sound in the medium.

Furthermore, for the sake of definiteness, we will consider that the random inhomogeneities are only in the right half-space $(x > 0)$. The left half-space $(x < 0)$ contains no random inhomogeneities. The equation of the incident plane wave in the left half-space can be given as

$$\psi_o = A_o e^{i(\omega t - kx)}$$

The wave equation in the right half-space will be

$$\psi = A(\vec{r})e^{i[\omega t - S(\vec{r})]} \tag{3}$$

If the usual method of small perturbations be used in looking for ψ, which is often applied in scatter theory taking the zero approximation ψ_o into account, then the expressions obtained for the amplitude and phase fluctuations will be limited by the smallness requirements and, therefore, by small distances, since the fluctuations increase with distance. Consequently, it is expedient to use the small perturbations method in the form given by S. M. Rytov [3].

As we will see below, the small perturbations method in this form is not limited by such rigid requirements as is essential for the comparison of theory and experiment since large amplitude and phase fluctuations are often observed in experiment. The substance of the method is the replacement of the wave function ψ by another function φ which is related to the first through:

$$\psi = A_o e^{i[\omega t - \varphi(\vec{r})]} \tag{4}$$

From a comparison of (4) and (3) there follows that $\varphi(\vec{r})$ is determined by the following relation:

$$\varphi(\vec{r}) = S(\vec{r}) + i \ln\frac{A}{A_o} \tag{5}$$

As is seen, the real and imaginary parts of the function $\varphi(\vec{r})$ introduced determine the phase and the logarithm of the ratio of the amplitudes (level of sound intensity), respectively. We are directly concerned with the fluctuations of these quantities.

Replacing, on the basis of (4), ψ through φ in (2), we obtain an equation in φ :

(6) $$(\nabla\varphi)^2 + i\nabla^2\varphi = k^2n^2 \quad ; \quad k = \frac{\omega_o}{c_o}$$

The zero approximation φ_o satisfies the equation for the homogeneous medium:

(7) $$(\nabla\varphi_o)^2 + i\,\nabla^2\varphi_o = k^2$$

Subtracting (7) from (6), we obtain an equation for $\varphi' = \varphi - \varphi_o$:

$$2(\nabla\varphi_o\nabla\varphi') + i\nabla^2\varphi' = 2\mu k^2 + \left[\mu^2k^2 - (\nabla\varphi')^2\right]$$

Using the small perturbations method and assuming that $\nabla\varphi'$ (more accurately, the nondimensional quantity $\frac{1}{k}\nabla\varphi'$) has order μ , we discard the terms enclosed in the square brackets as second order quantities in μ . Then we obtain a linear equation in φ' :

(8) $$2(\nabla\varphi_o\nabla\varphi') + i\nabla^2\varphi' = 2\mu k^2$$

This equation was obtained under the assumption that

(9) $$\frac{1}{k}\nabla\varphi' \sim \mu_1 \quad \text{or} \quad \frac{1}{k}|\nabla\varphi'| << 1$$

This last condition means that the phase variation and the relative amplitude variation within a wave length must be small. Relation (9) imposes no limitations on the total variation of these quantities. Assuming $\varphi_o = kx$ in (8), we obtain:

(10) $$2k\frac{\partial\varphi'}{\partial x} + i\nabla^2\varphi' = 2\mu k^2$$

The exact solution of this equation is:

$$\varphi'(x,y,z) = \frac{ik^2}{2\pi} \iiint_V \frac{e^{-ik[r-(x-\xi)]}}{r} \mu(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta \tag{11}$$

where V is the region occupied by the random inhomogeneities.

Separating into real and imaginary parts, we obtain the following formulas for the phase and amplitude fluctuations:

$$S' = \frac{k^2}{2\pi} \iiint_V \frac{1}{r} \sin k[r-(x-\xi)]\, \mu(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta \tag{12}$$

$$\ln \frac{A}{A_0} = \frac{k^2}{2\pi} \iiint_V \frac{1}{r} \cos k[r-(x-\xi)]\, \mu(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta \tag{13}$$

In the rough-scale inhomogeneity case when $ka \gg 1$ (where a is the scale of the inhomogeneity), the scattering angles are small and do not exceed $\frac{1}{ka}$ in order of magnitude. For this reason, a substantial effect will be given by those inhomogeneities concentrated within the cone with vertex at the receiver and vertex angle of order $\frac{1}{ka}$. Within this cone, the formula $r = \sqrt{(x-\xi)^2 + \rho^2}$ where $\rho^2 = (y-\eta)^2 + (z-\zeta)^2$, can be replaced by the approximation

$$r \approx (x-\xi) + \frac{1}{2}\frac{\rho^2}{x-\xi} \tag{14}$$

Replacing $[r-(x-\xi)]$ in (12) and (13) by the approximation (14) and the quantity $\frac{1}{r}$ by $\frac{1}{x-\xi}$, we obtain:

$$S = \frac{k^2}{2\pi} \int_V \frac{\sin \frac{k\rho^2}{2(x-\xi)}}{x-\xi}\, \mu\, dv \tag{15}$$

$$\ln \frac{A}{A_0} = \frac{k^2}{2\pi} \int_V \frac{\cos \frac{k\rho^2}{2(x-\xi)}}{x-\xi}\, \mu\, dv \tag{16}$$

Let us introduce the new notation, assuming:

$$\Phi_1\left(\frac{x-\xi}{k},\rho\right) = \frac{k}{2\pi(x-\xi)} \sin \frac{k\rho^2}{2(x-\xi)}$$

$$\Phi_2\left(\frac{x-\xi}{k},\rho\right) = \frac{k}{2\pi(x-\xi)} \cos \frac{k\rho^2}{2(x-\xi)}$$

Then (15) and (16) are rewritten thus:

$$S'(x,y,z) = k \int_0^x \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_1\left(\frac{x-\xi}{k},\rho\right) \mu(\xi,\eta,\zeta)\ d\xi\ d\eta\ d\zeta \tag{17}$$

$$\ln \frac{A(x,y,z)}{A_o} = k \int_0^x \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_2\left(\frac{x-\xi}{k},\rho\right) \mu(\xi,\eta,\zeta)\ d\xi\ d\eta\ d\zeta \tag{18}$$

Therefore, having obtained the basic initial formulas, let us turn to the statistical consideration of the question. Let us characterize the statistical properties of the medium by the correlation function $R(r)$:

$$R(r) = \overline{\mu_1 \mu_2}$$

where μ_1 and μ_2 are small deviations of the index of refraction from the average value at two receiver points a distance r apart. The bar denotes the statistical average.

Formulas (17) and (18) can be used to explain the question of the correlation between the amplitude and phase fluctuations at the reception point $(L,0,0)$. To this end, let us multiply (17) by (18) and let us take the average and we obtain, after introducing the nondimensional variables $\xi' = k\xi$; $\eta' = k\eta$; $\zeta' = k\zeta$; $\rho' = k\rho$; $L' = kL$; $r' = kr$:

$$\left[S' \ln \frac{A}{A_o}\right]_{(L,0,0)} = \frac{1}{2} \int_0^{L'} \int_0^{L'} \iiiint_{-\infty}^{\infty} \Phi_1(L'-\xi_1',\rho_1')\ \Phi_2(L'-\xi_2',\rho_2')\ R(r')\ \times$$
$$\times d\xi_1'\ d\eta_1'\ d\zeta_1'\ d\xi_2'\ d\eta_2'\ d\zeta_2' \tag{19}$$

Transforming to relative coordinates $\eta = \eta_1' - \eta_2'$; $\zeta = \zeta_1' - \zeta_2'$ and the coordinates of the center of gravity $y = \frac{1}{2}(\eta_1' + \eta_2')$; $z = \frac{1}{2}(\zeta_1' + \zeta_2')$, we can integrate over y and z . Then, we obtain:

$$\left[S' \ln \frac{A}{A_o}\right]_{(L,0,0)} = \frac{1}{2} \int_0^{L'} \int_0^{L'} \iint_{-\infty}^{\infty} \Phi_2\left[2L'-(\xi_1'+\xi_2'),\rho\right] R(r')\ d\eta\, d\zeta\ d\xi_1'\ d\xi_2'$$
$$- \frac{1}{2} \int_0^{L'} \int_0^{L'} \iint_{-\infty}^{\infty} \Phi_2(\xi_1'-\xi_2',\rho) R(r')\ d\eta\, d\zeta\ d\xi_1'\ d\xi_2' \tag{20}$$

where $\rho^2 = \eta^2 + \zeta^2$. Formula (20) yields the general solution of the formulated problem. In order to use it, it is necessary to give the correlation function $R(r')$. For example, if we assume that it is:

$$R(r') = R_o \exp\left\{-\frac{r^2}{a^2}\right\} = R_o \exp\left\{-\frac{r'^2}{k^2a^2}\right\} \tag{21}$$

where $r'^2 = \xi^2 + \eta^2 + \zeta^2$; $\xi = \xi_1' - \xi_2'$ then after integration in (20) we obtain:

$$\left[S' \ln \frac{A}{A_o}\right]_{(L,0,0)} = \frac{\sqrt{\pi}}{16} R_o a^3 k^3 \ln\left[1 + \left(\frac{4L}{ka^2}\right)^2\right] \tag{22}$$

and for the correlation coefficient R_{as}, we obtain on the basis of (22),(20) and (21):

$$R_{as} = \frac{1}{2} \frac{\ln (1 + D^2)}{\sqrt{D^2 - (\arctan D)^2}} \tag{23}$$

where $D = \frac{4L}{ka^2}$. At small distances $(D \ll 1)$ when geometric optics is suitable, (23) yields:*

$$R_{as} \cong \frac{1}{2}\sqrt{\frac{3}{2}} \cong 0.6$$

At large distances $(D \gg 1)$, (23) becomes:

$$R_{as} = \frac{\ln D}{D}$$

i.e., the correlation coefficient decreases with distance, approaching zero. Therefore, the correlation between the amplitude and phase fluctuations is substantial at short distances and vanishes at long.

Now, let us analyze the question of the auto-correlation of the amplitude and phase fluctuations at various reception points.

Let us assume that both receivers lie in the $x = L$ plane separated by a distance l.

The receiver coordinates are $(L,0,0)$ and $(L,0,l)$, respectively. The amplitude

---

* The author erroneously gave half this result in his paper (DAN,98,No.6,1954).

and phase fluctuations are determined by (17) and (18) both for the first and second receivers. The difference is only that $\rho_1^2 = \eta_1^2 + \zeta_1^2$ for the first and that $\rho_2^2 = \eta_2^2 + (l - \zeta_2)^2$ for the second. The auto-correlation functions for the phases and amplitudes are determined, consequently, by the following:

$$\overline{S_1'S_2'} = \frac{1}{2}(I_1 + I_2) \tag{24}$$

$$\overline{\ln\frac{A_1}{A_0}\ln\frac{A_2}{A_0}} = \frac{1}{2}(I_1 - I_2) \tag{25}$$

where

$$I_1 = \int_0^{L'}\int_0^{L'}\iint_{-\infty}^{\infty} \Phi_1(\xi_1' - \xi_2', \rho)\, R(r')\, d\eta\, d\zeta\, d\xi_1'\, d\xi_2' \tag{26}$$

$$I_2 = \int_0^{L'}\int_0^{L'}\iint_{-\infty}^{\infty} \Phi_1[2L - (\xi_1' + \xi_2'), \rho]\, R(r')\, d\eta\, d\zeta\, d\xi_1'\, d\xi_2' \tag{27}$$

$$\rho^2 = \eta^2 + (\zeta + l')^2 \; ; \quad l' = kl$$

Formulas (24) - (27) give the solution of the problem of the amplitude and phase auto-correlation at various reception points in general form. Assuming that the correlation function $R(r')$ is given by (21), and integrating in (26) and (27) by assuming that $ka \gg 1$, $L \gg a$, we obtain:

$$I_1 = \sqrt{\pi}\, R_0 a k^2 L \exp\left\{-\frac{l^2}{a^2} - \left(\frac{l^2}{a^2}\frac{1}{ka}\right)^2\right\} \tag{28}$$

$$I_2 = \frac{\sqrt{\pi}}{8i} R_0 a^3 k^3 [Ei(\varepsilon_2) - Ei(\varepsilon_1)] \tag{29}$$

where $\varepsilon_2 = -\frac{i\frac{l^2}{a^2}}{D + i}$, $\varepsilon_1 = \frac{i\frac{l^2}{a^2}}{D - i}$, Ei is the exponential integral. In the particular case, $l = 0$; hence there result the mean square amplitude and phase fluctuation formulas obtained by A. M. Obukhov [2].

$$\overline{S'^2} = \frac{\sqrt{\pi}}{2} R_0 a k^2 L \left(1 + \frac{1}{D}\arctan D\right) \tag{30}$$

$$\overline{\left(\ln\frac{A}{A_0}\right)^2} = \frac{\sqrt{\pi}}{2} R_0 a k^2 L \left(1 - \frac{1}{D}\arctan D\right) \tag{31}$$

From (24),(25), (28), (29), (30) and (31) we find the following expression for the auto-correlation coefficient:

$$R_{a,s} = \frac{\exp\left\{-\frac{l^2}{a^2} - \left(\frac{l^2}{a^2}\frac{1}{ka}\right)^2\right\} \pm \frac{1}{2iD}\left[\mathrm{Ei}(\varepsilon_2) - \mathrm{Ei}(\varepsilon_1)\right]}{1 \pm \frac{1}{D}\arctan D} \tag{32}$$

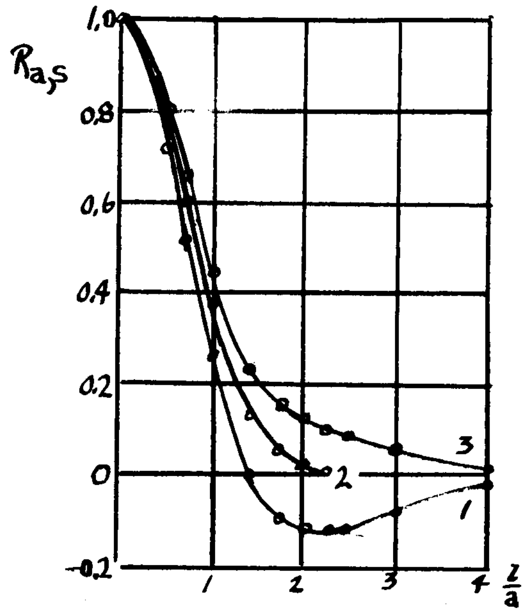

Figure 1. 1 - Auto-correlation coefficient for amplitude fluctuations; 2 - correlation coefficient for the index of refraction fluctuations; 3 - auto-correlation coefficient for phase fluctuations.

where the upper sign should be taken in calculating the phase auto-correlation function, R_s, and the lower for the amplitude auto-correlation function, R_a. As seen from (32), the auto-correlation function depends only on the three nondimensional parameters: $\frac{l}{a}$, D, ka. For small and average distances ($D \lesssim 1$) we obtain an asymptotic value of $R_{a,s}$ for large enough $\frac{l}{a}$, if we use the asymptotic expression for the Ei function [7]:

$$R_{a,s} \cong \frac{\exp\left\{-\frac{l^2}{a^2} - \left(\frac{l^2}{a^2}\frac{1}{ka}\right)^2\right\}}{1 \pm \frac{1}{D}\arctan D} \pm \frac{\left\{\frac{\sin\frac{l^2}{a^2}\frac{D}{1+D^2}}{\frac{l^2}{a^2}D} + \frac{\cos\frac{l^2}{a^2}\frac{D}{1+D^2}}{\frac{l^2}{a^2}}\right\}\exp\left(-\frac{l^2}{a^2}\frac{1}{1+D^2}\right)}{1 \pm \frac{1}{D}\arctan D} \tag{33}$$

The auto-correlation coefficient is small in comparison with unity because of the exponential factor in the numerator. This means that the correlation between the amplitude and phase fluctuations extends over a distance l of the order of the radius of correlation a in the medium.

For long distances $(D \gg 1)$, (32) becomes:

$$R_{a,s} = \frac{\exp\left\{-\frac{l^2}{a^2} - \left(\frac{l^2}{a^2}\frac{1}{ka}\right)^2\right\} \pm \frac{1}{D}\left[\frac{\pi}{2} - \mathrm{Si}\left(\frac{1}{D}\frac{l^2}{a^2}\right)\right]}{1 \pm \frac{1}{D}\arctan D} \qquad (34)$$

The dependence of the auto-correlation coefficient on the distance l between the receivers is shown graphically (fig. 1) for the case of $D = 10$, $ka = 10^2 - 10^4$. The middle curve is the correlation coefficient for the index of refraction as given by (21). From the graph, it can be seen that the auto-correlation between the amplitude and phase fluctuations extends a distance of the order of the radius of correlation of the inhomogeneities in the medium. We arrived at the same conclusion when we considered short and average distances.

I wish to express deep gratitude to L. M. Brekhovskikh for valuable suggestions.

Acoustics Institute, AN USSR
Moscow

October, 1954

References

1. B. A. KRASIL'NIKOV: Izvestia, AN USSR, geogr. and geophys. ser., No. 1, 33, 1949
2. A. M. OBUKHOV: Izvestia, AN USSR, geophys. ser., No. 2, 155, 1953
3. S. M. RYTOV: Izvestia, AN USSR, phys. ser., No. 2, 223, 1937
4. P. G. BERGMANN: Phys. Rev., 70, No. 7-8, 1946
5. D. I. MINZER: Acoust. Soc. Am., 25, 5, 922, 1953; 25, 6,1107, 1953
6. C. L. PEKERIS: Phys. Rev., 71, 268, 1947
7. I. M. RYZHIK, I. S. GRADSHTEIN: Tables of integrals, sums, series and products. Gostekhteorizdat, 1951, p. 324